

RODRIGUES FORMULAS FOR THE MACDONALD POLYNOMIALS

LUC LAPOINTE AND LUC VINET

ABSTRACT. We present formulas of Rodrigues type giving the Macdonald polynomials for arbitrary partitions λ through the repeated application of creation operators B_k , $k = 1, \dots, \ell(\lambda)$ on the constant 1. Three expressions for the creation operators are derived one from the other. When the last of these expressions is used, the associated Rodrigues formula readily implies the integrality of the (q, t) -Kostka coefficients. The proofs given in this paper rely on the connection between affine Hecke algebras and Macdonald polynomials.

1. INTRODUCTION

The Macdonald polynomials $J_\lambda(x; q, t)$ form a two-parameter basis for symmetric polynomials [1]. They play an important role in algebraic combinatorics and, in mathematical physics, they occur in particular in the wave functions of integrable quantum many-body models [2]. We shall show that the polynomials $J_\lambda(x)$ in N variables can be constructed by acting with a string of creation operators B_k , $k = 1, \dots, N$ on the constant 1, and shall thereby give Rodrigues formulas for these polynomials. Such results were first obtained in [3] in the limit case $q = t^\alpha$, $t \rightarrow 1$ of the Jack polynomials and proved rather useful [4].

Three expressions $B_k^{(i)}$ $i = 1, 2, 3$, will be obtained for the creation operators of the Macdonald polynomials. Expression $B_k^{(1)}$ will first be derived using the Pieri formula. The operator $B_k^{(2)}$ will then be shown to be equal to the operator $B_k^{(1)}$ and expression $B_k^{(3)}$ will finally be obtained from $B_k^{(2)}$ by observing that many terms in $B_k^{(2)}$ (and hence in $B_k^{(1)}$) act trivially on the Macdonald polynomials J_λ associated to partitions with no more than k parts. Expression $B_k^{(1)}$ was first derived in [5] where in addition, the q -difference operator version of $B_k^{(3)}$ was given as a conjecture. This third expression was also found by Kirillov and Noumi who provided two proofs [6, 7] of the fact that the operators $B_k^{(3)}$ are creation operators for the Macdonald polynomials. It should be pointed out that the integrality of the (q, t) -Kostka coefficients¹ is an immediate consequence of the Rodrigues formula for $J_\lambda(x)$ associated to $B_k^{(3)}$. We shall derive this formula from the one involving the operators $B_k^{(1)}$ by obtaining, as an intermediate step, the Rodrigues formula with the $B_k^{(2)}$ as creation operators. Our proofs will rely in an essential way on the connection between affine Hecke algebras and Macdonald polynomials [8, 9]. They will use in particular the fact that the Macdonald operators can be realized in terms of Dunkl-Cherednik operators. This is the main difference between the

¹Other proofs of the integrality !! of the (q, t) -Kostka coefficient

ts have been given recently using different approaches by Garsia and Remmel [11], Garsia and Tesler [12], Knop [13, 14] and Sahi [15].

proofs presented here and the short derivations given in [10]. It should be stressed that contrary to the approach followed in [10], the one taken here makes it possible to arrive at $B_k^{(2)}$ in a constructive fashion.

The outline of the paper is as follows. In Section 2, basic facts about the relevant affine Hecke algebra realization are collected. The Macdonald polynomials are introduced in Section 3 together with the commuting operators of which they are the simultaneous eigenfunctions. Section 4 is the bulk of the paper. This is where the expressions $B_k^{(1)}$, $B_k^{(2)}$ and $B_k^{(3)}$ are given (see (31),(32) and (34)) and derived (see Theorems 7,10 and 15).

2. THE AFFINE HECKE ALGEBRA $H(\tilde{W})$ [6, 8, 9]

Let $\Lambda_N = \mathbb{Q}(q, t)[x_1, \dots, x_N]$ be the ring of polynomials in the N variables x_1, \dots, x_N with coefficients in $\mathbb{Q}(q, t)$, the field of rational functions in the two indeterminates q and t . The Weyl group $W \cong S_N$ is generated by the transpositions $s_i, i = 1, \dots, N$. On $x^\lambda = x_1^{\lambda_1} \dots x_N^{\lambda_N}$ their action is such that

$$s_i x^\lambda = x^{s_i \lambda} s_i, \quad (1)$$

where $s_i \lambda = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \dots, \lambda_N)$. We denote by Λ_N^W , the subring of all symmetric polynomials. We can extend the action of the Weyl group W on Λ_N to one of the affine Weyl group \tilde{W} by introducing the elements s_0 and $\omega^{\pm 1}$ realized by:

$$\begin{aligned} s_0 &= s_{N-1} \dots s_2 s_1 s_2 \dots s_{N-1} \tau_1 \tau_N^{-1}, \\ \omega &= s_{N-1} \dots s_1 \tau_1 = \tau_N s_{N-1} \dots s_1. \end{aligned} \quad (2)$$

where τ_i , the shift operator, is such that

$$\tau_i f(x_1, \dots, x_N) = f(x_1, \dots, qx_i, \dots, x_N) \quad (3)$$

for any polynomial $f \in \Lambda_N$.

The generators of \tilde{W} obey the fundamental relations:

$$\begin{aligned} \text{(i)} \quad & s_i^2 = 1 & i = 0, 1, \dots, N-1, \\ \text{(ii)} \quad & s_i s_j = s_j s_i & |i - j| \geq 2, \\ \text{(iii)} \quad & s_i s_j s_i = s_j s_i s_j & |i - j| = 1, \\ \text{(iv)} \quad & \omega s_i = s_{i-1} \omega & i = 0, 1, \dots, N-1. \end{aligned} \quad (4)$$

where the indices $0, 1, \dots, N-1$ are understood as elements of $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. In the case of the Weyl group W , for any $w \in W$, there is a smallest positive integer p such that $w = s_{i_1} \dots s_{i_p}$ (reduced decomposition). We say that p is the length $L(w)$ of w . Let $v, w \in W$, in the Bruhat order, $v \leq w$ if v is of the form $v = s_{j_1} \dots s_{j_q}$ with (j_1, \dots, j_q) a subsequence of (i_1, \dots, i_p) .

The operators

$$T_i = 1 + \frac{1 - t^{-1} x_{i+1}/x_i}{1 - x_{i+1}/x_i} (s_i - 1), \quad (5)$$

for $i = 1, \dots, N-1$ and

$$T_0 = 1 + \frac{1 - t^{-1} q^{-1} x_1/x_N}{1 - q^{-1} x_1/x_N} (s_0 - 1), \quad (6)$$

and $\omega^{\pm 1}$ realize on Λ_N the affine Hecke algebra $H(\tilde{W})$ of \tilde{W} , that is they verify the defining relations

$$\begin{aligned} \text{(i)} \quad & (T_i - 1)(T_i + t^{-1}) = 0 & i = 0, 1, \dots, N-1, \\ \text{(ii)} \quad & T_i T_j = T_j T_i & |i - j| \geq 2, \\ \text{(iii)} \quad & T_i T_j T_i = T_j T_i T_j & |i - j| = 1, \\ \text{(iv)} \quad & \omega T_i = T_{i-1} \omega & i = 0, 1, \dots, N-1, \end{aligned} \tag{7}$$

where again the indices are understood as elements of $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$. The Dunkl-Cherednik operators Y_1, \dots, Y_N are constructed as follows from the generators of $H(\tilde{W})$:

$$Y_i = T_i \dots T_{N-1} \omega T_1^{-1} \dots T_{i-1}^{-1}. \tag{8}$$

They form an Abelian algebra: $[Y_i, Y_j] = 0, \forall i, j = 1, \dots, N$. They also satisfy the following commutation relations with the T_i 's:

$$\begin{aligned} T_i Y_{i+1} T_i &= Y_i, \\ T_i Y_j &= Y_j T_i & j \neq i, i+1. \end{aligned} \tag{9}$$

Let $J = \{j_1, j_2, \dots, j_\ell\}$ denote sets of cardinality $|J| = \ell$ made out of integers $j_\kappa \in \{1, \dots, N\}$, $1 \leq \kappa \leq \ell$ such that $j_1 < j_2 < \dots < j_\ell$. We introduce the operators

$$Y_{J,u} = (1 - ut^{\ell-1} Y_{j_1}) \dots (1 - ut Y_{j_{\ell-1}}) (1 - u Y_{j_\ell}), \tag{10}$$

associated to such sets and labelled by a real number u . If $|J| = 0$, we define $Y_{J,u} = 1$.

For each $w = s_{i_1} \dots s_{i_p} \in W$, T_w is defined by

$$T_w = T_{i_1} \dots T_{i_p}. \tag{11}$$

Note that T_w does not depend on the choice of the reduced decomposition of w .

The following relations between the generators of $H(\tilde{W})$ and the variables x_i will prove useful

$$\begin{aligned} T_i x_i &= x_{i+1} T_i - x_{i+1} (1 - t^{-1}), \\ T_i x_{i+1} &= x_i T_i + x_{i+1} (1 - t^{-1}), \\ T_i x_j &= x_j T_i & j \neq i, i+1, \\ T_i^{-1} x_i &= x_{i+1} T_i^{-1} + x_i (1 - t), \\ T_i^{-1} x_{i+1} &= x_i T_i^{-1} - x_i (1 - t), \\ \omega x_i &= x_{i-1} \omega & i \neq 1, \\ \omega x_1 &= q x_N \omega. \end{aligned} \tag{12}$$

From (7) and (12), we see that the x_i 's, the T_j 's and $\omega^{\pm 1}$ form an algebra over the field $\mathbb{Q}(q, t)$. An element O of this algebra will be said to be normally ordered if all the variables x_i 's have been moved to the left, that is if O has been put in the form

$$O = \sum_{\lambda} x^{\lambda} O_{\lambda}, \tag{13}$$

where O_{λ} is in $\mathbb{Q}(q, t)[T_i, \omega^{\pm 1}]$.

3. THE MACDONALD POLYNOMIALS [1]

Let $\lambda \in P \equiv \mathbb{N}^N$. We denote by $|\lambda| = \sum_i \lambda_i$, the degree of λ , and by $\ell(\lambda)$ the number of non-zero entries in λ . The dominance order on the set $P^+ \subseteq P$ of all partitions $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$, is $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $\lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i$ for all i . This ordering is extended to P as follows [16]. The orbit $W\lambda$ of $\lambda \in P$ under the action of the symmetric group $W \cong S_N$ will contain a unique partition $\lambda^+ \in P^+$. We denote by w_λ , the unique element of minimal length such that $w_\lambda \lambda = \lambda^+$. We have $\lambda \geq \mu$ if either $\lambda^+ > \mu^+$ or $\lambda^+ = \mu^+$ and $w_\lambda \leq w_\mu$ in the Bruhat order of W . Note that λ^+ is the unique maximum of $W\lambda$.

Homogeneous symmetric polynomials are labelled by partition λ of their degree. In the remainder of this section, λ always stands for a partition, that is $\lambda \in P^+$.

Three standard bases for Λ_N^W , the space of symmetric functions, are:

- (i) the power sum symmetric functions p_λ which in terms of the power sums

$$p_i = \sum_k x_k^i, \quad (14)$$

are given by

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots, \quad (15)$$

- (ii) the monomial symmetric functions m_λ which are

$$m_\lambda = \sum_{\text{distinct permutations}} x_1^{\lambda_1} x_2^{\lambda_2} \dots \quad (16)$$

- (iii) the elementary symmetric functions e_λ which in terms of the i^{th} elementary function

$$e_i = \sum_{j_1 < j_2 < \dots < j_i} x_{j_1} x_{j_2} \dots x_{j_i} = m_{(1^i)}, \quad (17)$$

are given by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots \quad (18)$$

The Macdonald polynomials can now be presented as follows. To the partition λ with $m_i(\lambda)$ parts equal to i , we associate the number

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \dots \quad (19)$$

We define a scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on Λ_N^W by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \quad (20)$$

where $\ell(\lambda)$ is the number of parts of λ . The Macdonald polynomials $J_\lambda(x; q, t) \in \Lambda_N^W$ are uniquely specified by

$$(i) \langle J_\lambda, J_\mu \rangle_{q,t} = 0, \quad \text{if } \lambda \neq \mu, \quad (21)$$

$$(ii) J_\lambda = \sum_{\mu \leq \lambda} v_{\lambda\mu}(q, t) m_\mu, \quad (22)$$

$$(iii) v_{\lambda\lambda}(q, t) = c_\lambda(q, t), \quad (23)$$

where

$$c_\lambda(q, t) = \prod_{s \in \lambda} (1 - q^{a(s)} t^{\ell(s)+1}). \quad (24)$$

As usual $a(s)$ and $\ell(s)$ denote the number of squares in the diagram associated to the partition λ that are respectively to the south and east of the square s .

For $r = 1, \dots, N$, let M_N^r denote the Macdonald operator

$$M_N^r = \sum_I t^{(N-r)r+r(r-1)/2} \tilde{A}_I(x; t) \prod_{i \in I} \tau_i, \quad (25)$$

where the sum is over all r -element subsets I of $\{1, \dots, N\}$,

$$\tilde{A}_I(x; t) = \prod_{\substack{i \in I \\ j \in I^c}} \frac{x_i - t^{-1}x_j}{x_i - x_j}, \quad (26)$$

and $M_N^0 \equiv 1$. These operators commute with each other, $[M_N^r, M_N^l] = 0$ and are diagonal on the Macdonald polynomials basis. From the Macdonald operators, one constructs

$$M_N(X; q, t) = \sum_{r=0}^N M_N^r X^r, \quad (27)$$

with X an arbitrary parameter. With J , a set of cardinality $|J| = j$, we shall also use $M_J(X; q, t)$ to represent the operator $M_j(X; q, t)$ in the variables x_i , $i \in J$. The generating function M_N will play a crucial role in the following. Its action on $J_\lambda(x; q, t)$ with $\ell(\lambda) \leq N$ is given, remarkably, by

$$M_N(X; q, t) J_\lambda(x; q, t) = a_\lambda(X; q, t) J_\lambda(x; q, t), \quad (28)$$

where

$$a_\lambda(X; q, t) = \prod_{i=1}^N (1 + X q^{\lambda_i} t^{N-i}). \quad (29)$$

From (27) we see that the eigenvalue of M_N^r on $J_\lambda(x; q, t)$ is the coefficient of X^r in the polynomial (29).

It is known (see for instance [6]) that the Macdonald operators can be rewritten in terms of Dunkl-Cherednik operators on Λ_N^W . In particular, we have that

$$\text{Res } Y_{\{1, \dots, N\}, u} = M_N(-u; q, t), \quad (30)$$

where $\text{Res } O$ means that O is restricted to act on Λ_N^W .

4. CREATION OPERATORS

We now give the expressions $B_k^{(i)}$ $i = 1, 2, 3$; $k = 1, \dots, N$ of the creation operators that we will derive in the remainder of the paper.

- Expression 1

$$B_k^{(1)} = \frac{1}{(q^{-1}; t^{-1})_{N-k}} Y_{\{1, \dots, N\}, t^{k+1-N} q^{-1} e_k}, \quad (31)$$

where for n positive integer, $(a; q)_n = (1-a)(1-qa) \dots (1-q^{n-1}a)$ and $(a; q)_0 \equiv 1$.

• Expression 2

$$B_k^{(2)} = \sum_{|I|=k} x_I \sum_{m=0}^{N-k} \sum_{\substack{I' \subseteq I^c \\ |I'|=m}} \frac{q^{-m}}{(t^{k-N+1}q^{-1}; t)_m} t^{-d(I', I^c)} Y_{I \cup I', t^{1-m}}. \quad (32)$$

The quantity $d(I, J)$ entering in the above expression depends on nested subsets $J \subseteq I$ of $\{1, \dots, N\}$ and is defined as follows. Order the elements of I so that $I = \{i_0, \dots, i_{\ell-1}\}$ with $i_0 < i_1 < \dots < i_{\ell-1}$. Let $J = \{i_{j_1}, \dots, i_{j_m}\} \subseteq I$ with its elements ordered so that $(i_{j_1}, \dots, i_{j_m})$ is a subsequence of $(i_0, \dots, i_{\ell-1})$, $0 \leq j_\kappa \leq \ell-1$; $k = 1, \dots, m$. We then define

$$d(J, I) = \sum_{k=1}^m j_k - m(m-1)/2. \quad (33)$$

Note that the sum is over the indices that identify the elements of J in the reference set I . If $|J| = 0$, then $d(J, I) \equiv 0$.

• Expression 3

$$B_k^{(3)} = \sum_{|I|=k} x_I Y_{I, t}. \quad (34)$$

We need to prove that these three sets of operators are such that

$$B_k^{(i)} J_\lambda(x) = J_{\lambda+(1^k)}(x), \quad (35)$$

if $\ell(\lambda) \leq k$. If this is so, the following Rodrigues formula for the Macdonald polynomials associated to any partition λ are easily seen to hold

$$J_\lambda(x; q, t) = (B_N^{(i)})^{\lambda_N} (B_{N-1}^{(i)})^{\lambda_{N-1}-\lambda_N} \dots (B_1^{(i)})^{\lambda_1-\lambda_2} \cdot 1. \quad (36)$$

That the first expression has property (35) will follow from the Pieri formula which gives the action of the elementary symmetric functions e_k on the monic Macdonald polynomials $P_\lambda = 1/c_\lambda(q, t) J_\lambda$. This formula reads [1]

$$e_k P_\lambda = \sum_{\mu} \Psi_{\mu/\lambda} P_\mu, \quad (37)$$

where the sum is over all partitions μ containing λ such that the set-theoretic difference $\mu - \lambda$ is k -dimensional with the property that $\mu_i - \lambda_i \leq 1$, $\forall i \geq 1$. With $C_{\mu/\lambda}$ and $R_{\mu/\lambda}$ respectively denoting the union of the columns and of the rows that intersect $\mu - \lambda$, the coefficients $\Psi_{\mu/\lambda}$ are given by

$$\Psi_{\mu/\lambda} = \prod_{\substack{s \in C_{\mu/\lambda} \\ s \notin R_{\mu/\lambda}}} \frac{b_\mu(s)}{b_\lambda(s)} \quad (38)$$

where

$$b_\lambda(s) = \begin{cases} \frac{1-q^{a(s)}t^{\ell(s)+1}}{1-q^{a(s)+1}t^{\ell(s)}} & \text{if } s \in \lambda \\ 1 & \text{otherwise} \end{cases}. \quad (39)$$

We shall then construct $B_k^{(2)}$ from $B_k^{(1)}$, using the realization of the affine Hecke algebra $H(\tilde{W})$ given in Section 2 and properties of the Dunkl-Cherednik operators. Upon proving the operator equality $B_k^{(2)} = B_k^{(1)}$ we shall infer that $B_k^{(2)}$ are indeed

creation operators. Last, we shall prove to conclude the derivation that $B_k^{(3)} J_\lambda = B_k^{(2)} J_\lambda = J_{\lambda+(1^k)}$ on Macdonald polynomials with $\ell(\lambda) \leq k$.

We start by giving some results that we will need in the sequel. First, a lemma that has to do with the normal ordering of some expressions (see(13)):

Lemma 1.

$$Y_k x_\ell = \sum_{j \geq \ell} x_j O_j, \quad (40)$$

with $O_j \in \mathbb{Q}(q, t)[T_i, \omega^{\pm 1}]$. And,

$$Y_k x_1 \dots x_\ell = \begin{cases} q x_1 \dots x_\ell Y_k & \text{if } \ell \geq k \\ x_1 \dots x_\ell Y_k & \text{if } \ell < k \end{cases} + \sum_{|\lambda|=\ell} x^\lambda O_\lambda, \quad (41)$$

where all λ 's in the sum contain at least one non-zero part λ_j with $j > \ell$, that is $\lambda \notin P^+$.

It is easily proved by induction from (12). A corollary of (40) is:

Corollary 2. For any $\lambda \in P$, $|\lambda| = \ell$, containing at least one non-zero part λ_j with $j > \ell$, we have, for any k ,

$$Y_k x^\lambda = \sum_{|\mu|=\ell} x^\mu O_\mu, \quad (42)$$

where all μ 's in the sum contain at least one non-zero part μ_j with $j > \ell$, that is $\mu \notin P^+$.

This is seen from (40) by commuting first Y_k with one of the x_j with $j > \ell$ and $\lambda_j \neq 0$.

Next a lemma about the normal ordering of expressions involving T_i and x^λ .

Lemma 3.

(i) if $s_i \lambda > \lambda$

$$T_i x^\lambda = x^{s_i \lambda} T_i + \sum_{\mu < s_i \lambda} x^\mu O_\mu, \quad (43)$$

(ii) if $s_i \lambda < \lambda$

$$T_i x^\lambda = \sum_{\mu < \lambda} x^\mu O_\mu, \quad (44)$$

(iii) if $s_i \lambda = \lambda$

$$T_i x^\lambda = x^\lambda T_i. \quad (45)$$

Proof. Since T_i commutes with all the variables except x_i and x_{i+1} , it suffices to look at the action of T_i on $x_i^{\lambda_i} x_{i+1}^{\lambda_{i+1}}$. The third case occurs when $\lambda_i = \lambda_{i+1}$ and it is trivially verified that $T_i(x_i x_{i+1})^{\lambda_i} = (x_i x_{i+1})^{\lambda_i} T_i$. From this result, in case (i) where $\lambda_{i+1} > \lambda_i$, we see upon factoring $(x_i x_{i+1})^{\lambda_i}$ that it suffices to consider the action of T_i on $x_{i+1}^{\lambda_{i+1}-\lambda_i}$. Similarly, in case (ii), we see that we only need to consider how T_i acts on $x_i^{\lambda_i-\lambda_{i+1}}$. The proof is then straightforwardly completed using (12).

Lemma 4. *If μ and λ with $\mu \neq \lambda$ are in the same orbit $W\lambda^+$ and such that $L(w_\mu) \geq L(w_\lambda) \neq 0$, then*

$$T_{w_\lambda} x^\mu = \sum_{\rho < \lambda^+} x^\rho O_\rho. \quad (46)$$

Proof. The only non-trivial case is when $L(w_\mu) = L(w_\lambda)$. In this case, we have from Lemma 3

$$T_{w_\lambda} x^\mu = T_{i_1} \dots T_{i_p} x^\mu = \sum_{\rho \leq w_{\mu,\lambda} \mu} x^\rho O_\rho, \quad (47)$$

with $w_{\mu,\lambda}$ some Weyl group element such that $w_{\mu,\lambda} < w_\lambda$ in the Bruhat order. In order to have $w_{\mu,\lambda} = w_\lambda$, case (i) of Lemma 3 would have to apply for every permutation s_{i_k} in the reduction of w_λ , but this is impossible since it would require that $s_{i_{p-k}}(s_{i_{p-k+1}} \dots s_{i_p} \mu) > s_{i_{p-k+1}} \dots s_{i_p} \mu$ for $k = 1, \dots, p-1$, in other words, it would demand that $w_\lambda \mu = \lambda^+$ which can not be the case because $\mu \neq \lambda$ by hypothesis. We thus have that all the ρ 's entering in (47) are such that $\rho \leq w_{\mu,\lambda} \mu < \lambda^+$, which proves the lemma.

Proposition 5. *If a non-zero operator is of the form $O = \sum_{|\mu|=k} x^\mu O_\mu$ with $O_\mu = 0$ when $\mu \in P^+$, there is at least one T_i , $i = 1, \dots, N-1$, for which $T_i O \neq O T_i$ and hence O is not symmetric.*

Proof. Suppose that O is symmetric and of the form $O = \sum_{|\mu|=k} x^\mu O_\mu$ with $O_\mu = 0$ when $\mu \in P^+$. There exists one term $x^\lambda O_\lambda$ of O with $O_\lambda \neq 0$ and such that, either $\mu^+ \not\prec \lambda^+$ or $\mu^+ = \lambda^+$ and $L(w_\mu) \not\prec L(w_\lambda)$, for all μ such that $O_\mu \neq 0$ in the decomposition of O . From Lemma 3 (which imply that $T_w x^\mu = \sum_{\rho \leq \mu^+} x^\rho \tilde{O}_\rho$ for any $w \in W$ and $\mu \in P$) and Lemma 4, we then have that

$$T_{w_\lambda} O = x^{\lambda^+} T_{w_\lambda} O_\lambda + \sum_{\rho; \rho \neq \lambda^+} x^\rho O'_\rho. \quad (48)$$

Since O is assumed to be symmetric, we also have

$$T_{w_\lambda} O = O T_{w_\lambda} = \sum_{\mu} x^\mu O''_\mu, \quad (49)$$

with $O''_\mu = 0$ if $\mu \in P^+$. From (48) and (49), since $T_{w_\lambda} O_\lambda \neq 0$, we have a contradiction and hence Proposition 5 must be true.

Corollary 6. *Two normally ordered symmetric operators A and B , whose factors of x^{λ^+} are the same for any $\lambda^+ \in P^+$, are equal.*

Since $A - B$ is symmetric and does not have any part in x^{λ^+} , $\forall \lambda^+ \in P^+$, it must be zero from Proposition 5.

Theorem 7. *For any partition λ with $\ell(\lambda) \leq k$, the operators $B_k^{(1)}$ act as follows on the Macdonald polynomials $J_\lambda(x; q, t)$:*

$$B_k^{(1)} J_\lambda(x) = J_{\lambda+(1^k)}(x). \quad (50)$$

Proof. The following lemma is an immediate consequence of the Pieri formula.

Lemma 8. *For λ a partition such that $\ell(\lambda) \leq k$, the action of e_k on P_λ is given by*

$$e_k P_\lambda = P_{\lambda+(1^k)} + \sum_{\mu \neq \lambda+(1^k)} \Psi_{\mu/\lambda} P_\mu, \quad (51)$$

where all the μ 's in the sum are such that $\mu_{k+1} = 1$.

Indeed, the only way to construct a μ with $\mu_{k+1} \neq 1$ is to add a 1 in each of the first k entries of λ . From Lemma 8 and (28) and (29), we have

$$Y_{\{1, \dots, N\}, t^{k+1-N} q^{-1}} e_k P_\lambda = \prod_{i=1}^k (1 - t^{k+1-i} q^{\lambda_i}) (q^{-1}; t^{-1})_{N-k} P_{\lambda+(1^k)} \quad (52)$$

since the eigenvalues

$$a_\mu(-t^{k+1-N} q^{-1}; q, t) = \prod_{i=1}^N (1 - t^{k+1-i} q^{\mu_i-1}), \quad (53)$$

of $Y_{\{1, \dots, N\}, t^{k+1-N} q^{-1}}$ on the P_μ 's in (51) vanish if $\mu_{k+1} = 1$.

From the definition given in (24), it is easy to check that

$$\frac{c_{\lambda+(1^k)}}{c_\lambda} = \prod_{i=1}^k (1 - t^{k+1-i} q^{\lambda_i}). \quad (54)$$

Using this result and passing from P_λ to J_λ we see that

$$B_k^{(1)} J_\lambda = \frac{1}{(q^{-1}; t^{-1})_{N-k}} Y_{\{1, \dots, N\}, t^{k+1-N} q^{-1}} e_k J_\lambda = J_{\lambda+(1^k)} \quad (55)$$

when $\ell(\lambda) \leq k$. This proves Theorem 7.

We are going to order $B_k^{(1)}$ normally and thus move all the variables contained in $e_k(x)$ to the left. In doing so, we shall only focus on terms having x^λ with $\lambda \in P^+$ on the left, knowing from Corollary 6, that we only need to symmetrize these terms in order to find the full expression. From Corollary 2, we see that the operators $Y_k x_I$ will not have terms of the form x^{λ^+} to the left whenever $I \neq \{1, \dots, k\}$. There thus only remains to consider

$$\begin{aligned} & \frac{1}{(q^{-1}; t^{-1})_{N-k}} (1 - t^k q^{-1} Y_1) \cdots (1 - t q^{-1} Y_k) \\ & \quad \times (1 - q^{-1} Y_{k+1}) \cdots (1 - t^{k+1-N} q^{-1} Y_N) x_1 \cdots x_k. \end{aligned} \quad (56)$$

From (41) and Corollary 2, we see that the expression below is the only term of type x^{λ^+} in (56) once the variables x_i 's have been moved to the left:

$$\begin{aligned} & \frac{1}{(q^{-1}; t^{-1})_{N-k}} x_1 \cdots x_k (1 - t^k Y_1) \cdots (1 - t Y_k) \\ & \quad \times (1 - q^{-1} Y_{k+1}) \cdots (1 - t^{k+1-N} q^{-1} Y_N). \end{aligned} \quad (57)$$

Thus,

$$\frac{1}{(q^{-1}; t^{-1})_{N-k}} x_1 \cdots x_k Y_{\{1, \dots, k\}, t} Y_{\{k+1, \dots, N\}, t^{k+1-N} q^{-1}} \quad (58)$$

is the only term of $B_k^{(1)}$ that has to the left a factor of the form x^{λ^+} . Before symmetrizing, we shall expand this last expression using the following lemma:

Lemma 9.

$$Y_{\{1, \dots, \ell\}, t^{-\ell+1}q^{-1}} = \sum_{m=0}^{\ell} q^{-m}(q^{-1}; t^{-1})_{\ell-m} \sum_{\substack{I \subseteq \{1, \dots, \ell\} \\ |I|=m}} t^{-d(I, \{1, \dots, \ell\})} Y_{I, t^{1-m}}. \quad (59)$$

Proof. We proceed by induction. The formula is easily seen to hold in the case $\ell = 1$. Assuming that (59) is true, we thus have

$$\begin{aligned} Y_{\{1, \dots, \ell+1\}, t^{-\ell}q^{-1}} &= Y_{\{1, \dots, \ell\}, t^{-\ell+1}q^{-1}} (1 - Y_{\ell+1} t^{-\ell} q^{-1}) \\ &= \sum_{m=0}^{\ell} q^{-m}(q^{-1}; t^{-1})_{\ell-m} \left(1 - t^{-(\ell-m)} q^{-1} + t^{-(\ell-m)} q^{-1} \right. \\ &\quad \left. - Y_{\ell+1} t^{-\ell} q^{-1} \right) \sum_{\substack{I \subseteq \{1, \dots, \ell\} \\ |I|=m}} t^{-d(I, \{1, \dots, \ell\})} Y_{I, t^{1-m}} \\ &= \sum_{m=0}^{\ell} q^{-m}(q^{-1}; t^{-1})_{\ell+1-m} \sum_{\substack{I \subseteq \{1, \dots, \ell+1\} \\ \ell+1 \notin I; |I|=m}} t^{-d(I, \{1, \dots, \ell+1\})} Y_{I, t^{1-m}} \\ &\quad + \sum_{m=0}^{\ell} q^{-m-1}(q^{-1}; t^{-1})_{\ell-m} \sum_{\substack{I \subseteq \{1, \dots, \ell+1\} \\ \ell+1 \in I; |I|=m+1}} t^{-d(I, \{1, \dots, \ell+1\})} Y_{I, t^{-m}} \\ &= \sum_{m=0}^{\ell+1} q^{-m}(q^{-1}; t^{-1})_{\ell+1-m} \sum_{\substack{I \subseteq \{1, \dots, \ell+1\} \\ |I|=m}} t^{-d(I, \{1, \dots, \ell+1\})} Y_{I, t^{1-m}}, \end{aligned} \quad (60)$$

which concludes the proof. In the derivation, we have used the following two properties of the quantity $d(J, I)$: $d(I, \{1, \dots, \ell\}) = d(I, \{1, \dots, \ell+1\})$ if $\ell+1 \notin I$ and $d(I \cup \{\ell+1\}, \{1, \dots, \ell+1\}) = d(I, \{1, \dots, \ell\}) + \ell - m$.

With the help of Lemma 9 and of the identity $Y_{\{1, \dots, k\}, t} Y_{I, t^{1-m}} = Y_{\{1, \dots, k\} \cup I, t^{1-m}}$ if $|I| = m$, expression (58) can be recast in the form

$$x_1 \dots x_k \sum_{m=0}^{N-k} \sum_{\substack{I \subseteq \{k+1, \dots, N\} \\ |I|=m}} \frac{q^{-m}}{(t^{k-N+1}q^{-1}; t)_m} t^{-d(I, \{k+1, \dots, N\})} Y_{\{1, \dots, k\} \cup I, t^{1-m}}. \quad (61)$$

We shall now give an expression in normal order (see (62)) which has (61) as its only term of type x^{λ^+} . By Corollary 6, in order to show that this expression coincides with $B_k^{(1)}$ we shall only need to prove that it is symmetric.

Theorem 10.

$$B_k^{(1)} = B_k^{(2)} \equiv \sum_{|I|=k} x_I \sum_{m=0}^{N-k} \sum_{\substack{I' \subseteq I^c \\ |I'|=m}} \frac{q^{-m}}{(t^{k-N+1}q^{-1}; t)_m} t^{-d(I', I^c)} Y_{I \cup I', t^{1-m}}, \quad (62)$$

hence $B_k^{(2)}$ is also such that $B_k^{(2)} J_{\lambda} = J_{\lambda+(1^k)}$ for $\ell(\lambda) \leq k$.

Proof. We first give some expressions that are easily checked to commute with T_i from properties (7),(9) and (12). With $f \in \Lambda_N^W$,

$$\begin{aligned}
\text{(I)} \quad & (T_i - 1)(x_i + x_{i+1})f = 0 \\
\text{(II)} \quad & (T_i - 1)x_i x_{i+1}f = 0 \\
\text{(III)} \quad & (T_i - 1)(1 - uY_i)(1 - uY_{i+1})f = 0 \\
\text{(IV)} \quad & (T_i - 1)\left[x_i(1 - uY_i) + x_{i+1}(1 - uY_{i+1})\right]f = 0 \\
\text{(V)} \quad & (T_i - 1)\left[(1 - uY_i) + t^{-1}(1 - uY_{i+1})\right]f = 0
\end{aligned} \tag{63}$$

That $B_k^{(2)}$ has expression (61) has its only term of type x^{λ^+} is obvious. In view of the remark made before Theorem 10, we now only need to verify that the operators $B_k^{(1)}$ as defined in (32) (and (62)) are symmetric, in other words that they satisfy $(T_i - 1)B_k^{(2)}f = 0$, $\forall i = 1, \dots, N-1$. Since two sets, I and I' , enter in the definition of $B_k^{(2)}$ we proceed by looking at all the inclusion possibilities of the indices i and $i+1$ into these two sets and then examine the particular terms in $B_k^{(2)}$ that are affected by the action of T_i . What we find using (63) is that these terms are separately or pairwise symmetric. Indeed consider the cases:

- (i) $i, i+1 \notin I$ and $i, i+1 \notin I'$: T_i trivially commutes.
- (ii) $i, i+1 \notin I$ and $i \in I', i+1 \notin I'$: add the case $\bar{I} = I, \bar{I}' = (I' \cup \{i+1\}) \setminus \{i\}$ which is such that $d(\bar{I}', \bar{I}^c) = d(I', I^c) + 1$. The symmetry follows from (V).
- (iii) $i, i+1 \notin I$ and $i, i+1 \in I'$: T_i commutes owing to (III).
- (iv) $i \in I, i+1 \notin I$ and $i+1 \notin I'$: add the case $\bar{I} = (I \cup \{i+1\}) \setminus \{i\}, \bar{I}' = I'$ which is such that $d(\bar{I}', \bar{I}^c) = d(I', I^c)$. The symmetry is verified with the help of (IV).
- (v) $i \in I, i+1 \notin I$ and $i+1 \in I'$: add the case $\bar{I} = (I \cup \{i+1\}) \setminus \{i\}, \bar{I}' = (I' \cup \{i\}) \setminus \{i+1\}$ which is such that $d(\bar{I}', \bar{I}^c) = d(I', I^c)$. The symmetry is then confirmed using (I) and (III).

All the other cases are immediate consequences of these cases and of (II). Theorem 10 is thus seen to hold.

When going from $B_k^{(2)}$ to $B_k^{(3)}$, it is useful to obtain $\text{Res } B_k^{(2)}$, that is the q -difference operator version of $B_k^{(2)}$. The next lemma gives the essential step.

Lemma 11.

$$\text{Res} \sum_{|I|=k} x_I \sum_{\substack{I' \subseteq I^c \\ |I'|=m}} t^{-d(I', I^c)} Y_{I \cup I', t^{1-m}} = \sum_{|I|=k} x_I \sum_{\substack{I' \subseteq I^c \\ |I'|=m}} \tilde{A}_{I \cup I'} M_{I \cup I'}(-t^{1-m}; q, t). \tag{64}$$

Proof. We first give two formulas that we will need. The first one is a well known identity and the second is a special case of a formula given by Garsia and Tesler ([12], Proposition 3.1).

Formula 12.

$$e_m(1, t^{-1}, \dots, t^{-N+1}) = t^{m(m-1)/2} t^{-m(N-1)} \left[\begin{matrix} N \\ m \end{matrix} \right]_t, \tag{65}$$

where the q -binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}. \quad (66)$$

Formula 13. With $N = |J \cup J^c|$, we have that, for any $k = 0, \dots, N$ and $m = 0, \dots, N - k$,

$$\sum_{|J|=k} x_J \sum_{\substack{J' \subseteq J^c \\ |J'|=m}} \tilde{A}_{J \cup J'} = t^{-m(N-k-m)} \begin{bmatrix} N-k \\ m \end{bmatrix}_t \sum_{|J|=k} x_J. \quad (67)$$

Given that

$$\sum_{\substack{|J'|=m \\ J' \subseteq J^c}} t^{-d(J', J^c)} = t^{m(m-1)/2} e_m(1, t^{-1}, \dots, t^{-(N-k)+1}), \quad (68)$$

for any subset J of $\{1, \dots, N\}$ of cardinality k , the following lemma follows from Formulas 12 and 13.

Lemma 14. With $N = |J \cup J^c|$, we have that, for any $k = 0, \dots, N$ and $m = 0, \dots, N - k$,

$$\sum_{|J|=k} x_J \sum_{\substack{|J'|=m \\ J' \subseteq J^c}} t^{-d(J', J^c)} = \sum_{|J|=k} x_J \sum_{\substack{|J'|=m \\ J' \subseteq J^c}} \tilde{A}_{J \cup J'}. \quad (69)$$

We now return to the proof of (64). Since both sides are symmetric, it will suffice to show that the coefficients of $\tau_1 \dots \tau_\ell$ for $\ell \leq N$ are identical on both sides of the equation.

To that end, let

$$\begin{aligned} I &= L \cup J, & I' &= \bar{L} \cup J', \\ L &\subseteq \{1, \dots, \ell\}, & \bar{L} &= \{1, \dots, \ell\} \setminus L, \\ J, J' &\subseteq \{1, \dots, N\} \setminus \{1, \dots, \ell\} = J \cup \bar{J}, \\ J \cap \bar{J} &= \phi, J' \subseteq \bar{J}, \end{aligned} \quad (70)$$

and define

$$[\ell, k] = \begin{cases} \ell & \text{if } \ell \leq k \\ k & \text{if } \ell > k \end{cases}. \quad (71)$$

The only place in the l.h.s. of (64) where the operator product $\tau_1 \dots \tau_\ell$ will occur is in $\text{Res } Y_1 \dots Y_\ell$ (see [6] Proposition 6.1 and formula 7.20). The coefficient of $\tau_1 \dots \tau_\ell$ in this expression is $\tilde{A}_{\{1, \dots, \ell\}}$. With this knowledge and the help of (10), (25) and (27), we find that the coefficients of $\tau_1 \dots \tau_\ell$ on both sides of (64) are respectively:

$$\begin{aligned} \text{l.h.s.} |_{\tau_1 \dots \tau_\ell} &= t^{\ell(\ell-1)/2} (-t^{k-\ell+1})^\ell \tilde{A}_{\{1, \dots, \ell\}} \\ &\quad \times \sum_{n=0}^{[\ell, k]} \sum_{|L|=n} x_L \sum_{|J|=k-n} x_J \sum_{|\bar{L} \cup J'|=m} t^{-d(J', \bar{J})} \end{aligned} \quad (72)$$

and

$$\begin{aligned} \text{r.h.s.}|_{\tau_1 \dots \tau_\ell} &= t^{\ell(\ell-1)/2} (-t^{1-m})^\ell t^{\ell(m+k-\ell)} \tilde{A}_{\{1, \dots, \ell\}} \\ &\times \sum_{n=0}^{[\ell, k]} \sum_{|L|=n} x_L \sum_{|J|=k-n} x_J \sum_{|\bar{L} \cup J'|=m} \tilde{A}_{J \cup J'}^{J \cup \bar{J}}, \end{aligned} \quad (73)$$

with

$$\tilde{A}_J^I = \prod_{\substack{i \in J \\ j \in I \setminus J}} \frac{x_i - t^{-1} x_j}{x_i - x_j}. \quad (74)$$

We have used in (72) the fact that $d(I', I^c) = d(J', \bar{J})$ and in (73), the identity $\tilde{A}_{I \cup I'} \tilde{A}_{\{1, \dots, \ell\}}^{I \cup I'} = \tilde{A}_{\{1, \dots, \ell\}} \tilde{A}_{I \cup I' \setminus \{1, \dots, \ell\}}^{\{1, \dots, N\} \setminus \{1, \dots, \ell\}}$. It is then immediate to see that the equality

$$\text{l.h.s. of (64)}|_{\tau_1 \dots \tau_\ell} = \text{r.h.s. of (64)}|_{\tau_1 \dots \tau_\ell} \quad (75)$$

holds, since after trivial simplifications it is seen to amount to

$$\begin{aligned} \sum_{n=0}^{[\ell, k]} \sum_{|L|=n} x_L \left(\sum_{|J|=k-n} x_J \sum_{\substack{|J'|=m-\ell+n \\ J' \subseteq \bar{J}}} t^{-d(J', \bar{J})} \right) = \\ \sum_{n=0}^{[\ell, k]} \sum_{|L|=n} x_L \left(\sum_{|J|=k-n} x_J \sum_{\substack{|J'|=m-\ell+n \\ J' \subseteq \bar{J}}} \tilde{A}_{J \cup J'}^{J \cup \bar{J}} \right) \end{aligned} \quad (76)$$

and hence to follow from Lemma 14.

Once Lemma 11 is proved, the connection between $B_k^{(2)}$ and $B_k^{(3)}$ is readily obtained.

Theorem 15. *For any partition λ , such that $\ell(\lambda) \leq k$, the actions of $B_k^{(2)}$ and $B_k^{(3)}$ on the Macdonald polynomials $J_\lambda(x)$ coincide:*

$$B_k^{(3)} J_\lambda(x) = B_k^{(2)} J_\lambda(x) = J_{\lambda+(1^k)}(x). \quad (77)$$

This is shown to be true with the help of the following lemma

Lemma 16. *Let $|I| = k$ and $|I'| = m$, $I' \subseteq I^c$.*

$$M_{I \cup I'}(-t^{1-m}; q, t) J_\lambda(x; q, t) = 0, \quad (78)$$

if $\ell(\lambda) \leq k$ and $m > 0$.

Proof. Denote by $x(I)$ the set of variables $\{x_i, i \in I\}$. The Macdonald polynomials are known [1] to enjoy the property according to which

$$J_\lambda(x(I), x(I^c)) = \sum_{\mu, \nu} \tilde{f}_{\mu\nu}^\lambda J_\mu(x(I)) J_\nu(x(I^c)) \quad (79)$$

with $\tilde{f}_{\mu\nu}^\lambda = 0$ unless $\mu \subset \lambda$ and $\nu \subset \lambda$ and in particular if $\ell(\mu)$ or $\ell(\nu)$ is greater than k .

Since $M_{I \cup I'}(-t^{1-m}; q, t)$ is a q -difference operator depending only of the variables x_i , $i \in I \cup I'$, we see from (79) that

$$M_{I \cup I'} J_\lambda(x) = \sum_{\mu, \nu} \tilde{f}_{\mu\nu}^\lambda J_\mu(x((I \cup I')^c)) M_{I \cup I'} J_\nu(x(I \cup I')). \quad (80)$$

The proof of Lemma 16 is then completed by observing from (28)-(29) that

$$M_{I \cup I'}(-t^{1-m}; q, t) J_\nu(x(I \cup I')) = \prod_{i=1}^{k+m} (1 - q^{\nu_i} t^{k+1-i}) J_\nu(x(I \cup I')) = 0, \quad (81)$$

whenever $m > 0$, since $\nu_{k+1} = 0$.

Theorem 15 is thus an immediate consequence of this lemma since, by (64), we have that:

$$\left(\sum_{|I|=k} x_I \sum_{\substack{I' \subseteq I^c \\ |I'|=m}} t^{-d(I', I^c)} Y_{I \cup I', t^{1-m}} \right) J_\lambda(x; q, t) = 0, \quad (82)$$

if $\ell(\lambda) \leq k$ and $m > 0$; this only leaves the $m = 0$ term of $B_k^{(2)}$, which coincides with $B_k^{(3)}$.

5. CONCLUSION

As already mentioned in the introduction, the integrality of the (q, t) -Kostka coefficients follows quite straightforwardly from the Rodrigues formula (36) when the operators $B_k^{(3)}$ are used as creation operators. This is explained in [6, 7, 10]. Other interesting properties of $B_k^{(3)}$ have been conjectured in [5]. In particular, a formula that would give the action of these operators on arbitrary Macdonald polynomials has been proposed: it looks like a deformation of the Pieri formula and, if true, would imply that the N operators $\text{Res} \sum_{|I|=m} x_I Y_{I, t^{\kappa-m+1}}$, $m = 1, \dots, N$ form a commuting set for any $\kappa \in \mathbb{R}$. We hope that the constructive approach presented in this paper will allow one to make progress towards proving these conjectures and unravelling the algebra of which the creation operators are part of.

Acknowledgments. We would like to express our thanks to Adriano Garsia for various comments and suggestions.

This work has been supported in part through funds provided by NSERC (Canada) and FCAR (Québec). L. Lapointe holds a NSERC postgraduate scholarship.

REFERENCES

- [1] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition, Clarendon Press, Oxford, 1995.
- [2] S. N. M. Ruijsenaars, *Complete integrability for the relativistic Calogero-Moser system and elliptic function identities*, Commun. Math. Phys. **110**, 191-213 (1987).
- [3] L. Lapointe and L. Vinet, *Exact operator solution of the Calogero-Sutherland model*, Commun. Math. Phys., to appear (1996).
- [4] L. Lapointe and L. Vinet, *A Rodrigues formula for the Jack polynomials and the Macdonald-Stanley conjecture*, IMRN **9**, 419-424 (1995).
- [5] L. Lapointe and L. Vinet, *Creation operators for the Macdonald and Jack polynomials*, Preprint CRM-2345.
- [6] A. Kirillov and M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, preprint.
- [7] A. Kirillov and M. Noumi, *q -Difference raising operators for Macdonald polynomials and the integrality of transitions coefficients*, preprint.

- [8] I. Cherednik, *Double affine Hecke algebras and Macdonald conjectures*, Annals of Math. **141**, 191-216 (1995).
- [9] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Séminaire Bourbaki, 47ème année, no. 797 (1994-95).
- [10] L. Lapointe and L. Vinet, *A short proof of the integrality of the Macdonald (q,t) -Kostka coefficients*, Preprint CRM-2360.
- [11] A.M. Garsia and J. Remmel, *Plethystic formulas and positivity for q,t -Kostka coefficients*, preprint.
- [12] A.M. Garsia and G. Tesler, *Plethystic formulas for Macdonald q,t -Kostka coefficients*, preprint.
- [13] F. Knop, *Integrality of two variable Kostka functions*, preprint.
- [14] F. Knop, *Symmetric and non-symmetric quantum Capelli polynomials*, preprint.
- [15] S. Sahi, *Interpolation, integrality and generalization of Macdonald polynomials*, IMRN (to appear).
- [16] S. Sahi, *A recursion and a combinatorial formula for the Jack polynomials*, preprint.

CENTRE DE RECHERCHES MATHÉMATIQUES,, UNIVERSITÉ DE MONTRÉAL, C.P. 6128, SUCCURSALE CENTRE-VILLE,, MONTRÉAL, QUÉBEC, CANADA, H3C 3J7